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Peripheric extended twists

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Abstract. The properties of the set $\widehat{\mathcal{L}}$ of extended Jordanian twists are studied. It is shown that the boundaries of $\widehat{\mathcal{L}}$ contain twists whose characteristics differ considerably from those of internal points. The extension multipliers of these 'peripheric' twists are factorizable. This leads to simplifications in the twisted algebra relations and helps to find the explicit form for coproducts. The peripheric twisted algebra $U(sl(4))$ is obtained to illustrate the construction. It is shown that the corresponding deformation $U_P(sl(4))$ cannot be connected with the Drinfeld–Jimbo one by a smooth limit procedure. All the carrier algebras for the extended and the peripheric extended twists are proved to be Frobenius.

1. Introduction

Any Lie bialgebra has a quantum deformation [1], although there are not many cases where it can be written in the global form. In this context explicit knowledge of the universal $\mathcal{R}$-matrix is of crucial importance. It provides the possibility of building the R-matrices in any representation and of using the advantages of the Fadeev–Reshetikin–Takhtajan (FRT) formalism [2]. This is why the triangular Hopf algebras and twists (they preserve the triangularity [3,4]) play such an important role in quantum group theory and applications [5–7]. Despite these facts very few types of twists were written explicitly in a closed form. The best known example is the Jordanian twist (JT) of $sl(2)$ or, more exactly, of its Borel subalgebra $B(2)$ ($\{H, E|[H, E] = 2E\}$) with $r = H \otimes E - E \otimes H = H \wedge E$ [8] where the triangular R-matrix $\mathcal{R} = (\mathcal{F}_j)_{21}\mathcal{F}_j^{-1}$ is defined by the twisting element [9, 10]

$$\mathcal{F}_j = \exp\{\tfrac{1}{2}H \otimes \ln(1 + 2\xi E)\}. \tag{1.1}$$

In [11] it was shown that there exist different extensions (ETs) of this twist. In particular, the ET deformation for $\mathcal{U}(sl(N))$ was constructed with the explicit expressions of deformed compositions. Using the notion of a factorizable twist [12] the element $\mathcal{F}_E \in \mathcal{U}(sl(N))^{\otimes 2}$

$$\mathcal{F}_E = \exp\left\{2\xi \sum_{i=2}^{N-1} E_{1i} \otimes E_{iN}e^{-\sigma}\right\} \exp\{H \otimes \sigma\} \tag{1.2}$$

was proved to satisfy the twist equation, where $E = E_{1N}$, $H = E_{11} - E_{NN}$ and $\sigma = \frac{1}{2}\ln(1 + 2\xi E)$. For simplicity of compositions the algebra $sl(N)$ is presented above in the standard $gl(N)$ basis, namely $\{E_{ij}\}_{i,j=1,\ldots,N}$.

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The connection of the Drinfeld–Jimbo (DJ) deformation [8, 13] with the Jordanian deformation had already been pointed out in [10]. The similarity transformation of the classical matrix

$$r_{\mathrm{DJ}} = \sum_{i=1}^{\mathrm{rank}(g)} t_{ij} H_i \otimes H_j + \sum_{\alpha\in\Delta_+} E_\alpha \otimes E_{-\alpha}$$

performed by the operator $\exp(\xi \text{ ad } E_{1N})$ (with the highest root generator E_{1N}) turns r_{DJ} into the sum $r_{\mathrm{DJ}} + \xi r_j$ [10] where

$$r_j = -\xi \left(H_{1N} \wedge E_{1N} + 2 \sum_{k=2}^{N-1} E_{1k} \wedge E_{kN} \right). \tag{1.3}$$

Hence r_j is also a classical r-matrix and defines the corresponding deformation. A contraction of the quantum Manin plane $xy = qyx$ of $\mathcal{U}_q(sl(2))$ with the above-mentioned similarity transformation in the fundamental representation $M = 1 + \theta\rho(E_{12})$, $\theta = \xi(1-q)^{-1}$ results in the Jordanian plane $x'y' = y'x' + \xi y'^2$ of $\mathcal{U}_j(sl(2))$ [9]. Thus, the Jordanian and the extended Jordanian twisted algebras (with the carrier subalgebra correlated with the standard dual $\mathfrak{g}^*_{\mathrm{DJ}}$) can be treated as a limiting case of the parameterized set of Drinfeld–Jimbo quantizations.

In this paper we study the family of carrier algebras (the term is considered to appear first in [10]) of the type $\mathbf{L}$, i.e. the three-parametric set $\mathcal{L} = \{\mathbf{L}(\alpha, \beta, \gamma, \delta)_{\alpha+\beta=\delta}\}$ and the properties of the corresponding sets $\widehat{\mathcal{L}}$ of twists when the parameters tend to its limiting values (section 3). We show that there are two cases ($\alpha \to 0$ and $\beta \to 0$) when the twists survive and remain non-trivial. We call these twists peripheric extended twists (PE twists or PETs); they form the boundary subsets of the $\widehat{\mathcal{L}}$ variety.

The properties of the peripheric algebras differ considerably from those of the internal points of $\mathcal{L}$. The same is true for the properties of PE twists. In contrast to the general situation, the extension factors of PE twists are the solutions of the factorized twist equations (see section 2). In section 4 we show how $\mathbf{L}(0, \beta, \gamma, \beta)$ or $\mathbf{L}(\alpha, 0, \gamma, \alpha)$ can be injected into the simple Lie algebras and illustrate all the results for the case $sl(4) \supset \mathbf{L}(-1, 0, 1, -1)$. The deformed coproducts thus obtained for $U_{\mathcal{F}_P}(sl(4))$ are much simpler than in the case of general ETs and the complete list of them for the generators of $sl(4)$ is presented. The other significant fact is that the PE twists cannot be connected with the DJ deformations by any kind of smooth 'contraction' (section 5). The solutions of the classical Yang–Baxter equation corresponding to ETs and PETs can easily be related with the classification given by Stolin [14]. The internal points of the $\mathcal{L}$ variety are Frobenius algebras. On the boundary only the above mentioned subsets $\{\mathbf{L}(0, \beta, \gamma, \beta)\}$ and $\{\mathbf{L}(\alpha, 0, \gamma, \alpha)\}$ are formed by Frobenius algebras. The paper is concluded by the discussion of relations between Drinfeld-Jimbo, extended twist and peripheric extended twist deformations.

2. Basic definitions

In this section we recall briefly the basic notions connected with the twisting procedure.

A Hopf algebra $\mathcal{A}(m, \Delta, \epsilon, S)$ with multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, coproduct $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $\epsilon: \mathcal{A} \to C$, and antipode $S : \mathcal{A} \to \mathcal{A}$ can be transformed [3] with an invertible (twisting) element $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$, $\mathcal{F} = \sum f_i^{(1)} \otimes f_i^{(2)}$, into a twisted one $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}})$. This Hopf algebra $\mathcal{A}_{\mathcal{F}}$ has the same multiplication and counit, but the twisted coproduct and antipode

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1} \qquad S_{\mathcal{F}}(a) = vS(a)v^{-1} \tag{2.1}$$

with

$$v = \sum f_i^{(1)} S(f_i^{(2)}) \qquad a \in \mathcal{A}.$$

The twisting element has to satisfy the equations

$$(\epsilon \otimes id)(\mathcal{F}) = (id \otimes \epsilon)(\mathcal{F}) = 1 \tag{2.2}$$

$$\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}). \tag{2.3}$$

The first is just a normalization condition, and follows from the second relation modulo a non-zero scalar factor.

If $\mathcal{A}$ is a Hopf subalgebra of $\mathcal{B}$ the twisting element $\mathcal{F}$ satisfying (2.1)–(2.3) induces the twist deformation $\mathcal{B}_{\mathcal{F}}$ of $\mathcal{B}$. In this case one can put $a \in \mathcal{B}$ in all expressions (2.1). This will completely define the Hopf algebra $\mathcal{B}_{\mathcal{F}}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the universal enveloping algebras: $\mathcal{A} = U(\mathfrak{l}) \subset \mathcal{B} = U(\mathfrak{g})$ with $\mathfrak{l} \subset \mathfrak{g}$. If $U(\mathfrak{l})$ is the minimal subalgebra on which $\mathcal{F}$ is completely defined as $\mathcal{F} \in U(\mathfrak{l}) \otimes U(\mathfrak{l})$ then $\mathfrak{l}$ is called the carrier algebra for $\mathcal{F}$ [10].

The composition of appropriate twists can be defined as $\mathcal{F} = \mathcal{F}_2\mathcal{F}_1$. Here the element $\mathcal{F}_1$ has to satisfy the twist equation with the coproduct of the original Hopf algebra, while $\mathcal{F}_2$ must be its solution for $\Delta_{\mathcal{F}_1}$ of the one twisted by $\mathcal{F}_1$. In particular, if $\mathcal{F}$ is a solution to the twist equation (2.3) then $\mathcal{F}^{-1}$ satisfies this equation with Δ substituted by $\Delta_{\mathcal{F}}$.

If the initial Hopf algebra $\mathcal{A}$ is quasitriangular with the universal element $\mathcal{R}$, then so is the twisted one $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ whose universal element is related to the initial $\mathcal{R}$ by a transformation

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\,\mathcal{R}\,\mathcal{F}^{-1}. \tag{2.4}$$

Most of the explicitly known twisting elements have the following factorization properties with respect to comultiplication:

$$(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}\mathcal{F}_{13} \qquad \text{or} \qquad (\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}$$

and

$$(id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13} \qquad \text{or} \qquad (id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{12}.$$

To guarantee the validity of the twist equation, these identities are to be combined with the additional requirement $\mathcal{F}_{12}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{12}$ or the Yang–Baxter equation on $\mathcal{F}$ [12].

An important subclass of factorizable twists consists of elements satisfying the equations

$$(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23} \tag{2.5}$$

$$(id \otimes \Delta_{\mathcal{F}})(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13}. \tag{2.6}$$

Apart from the universal R-matrix $\mathcal{R}$ that satisfies these equations for $\Delta_{\mathcal{F}} = \Delta^{op}$ ($\Delta^{op} = \tau \circ \Delta$, where $\tau(a \otimes b) = b \otimes a$) there are two more well developed cases of such twists: the Jordanian twist of a Borel algebra $B(2)$ where $\mathcal{F}_j$ has the form (1.1) (see [9]) with H being primitive in $B(2)$ and σ primitive in $\mathcal{U}_{\mathcal{F}_j}(B(2))$, and the extended Jordanian twist (see [11] for details).

It will be shown in section 3 that both sets of PE twists are not only factorizable but have factorizable extensions. One of these extensions satisfies the ordinary factorization equations (2.5) and (2.6), the other refers to a more sophisticated class.

According to the result of Drinfeld [4] skew (constant) solutions of the classical Yang–Baxter equation (CYBE) can be quantized, and the deformed algebras thus obtained can be presented in a form of twisted universal enveloping algebras. On the other hand, such solutions of CYBE can be connected with the quasi-Frobenius carrier subalgebras of the initial classical Lie algebra [14]. A Lie algebra $\mathfrak{g}(\mu)$, with the Lie composition μ, is called Frobenius if there exists a linear functional $g^* \in \mathfrak{g}^*$ such that the form $b(g_1, g_2) = g^*(\mu(g_1, g_2))$ is non-degenerate. This means that $\mathfrak{g}$ must have a non-degenerate 2-coboundary $b(g_1, g_2) \in B^2(\mathfrak{g}, \mathbf{K})$.

The algebra is called quasi-Frobenius if it has a non-degenerate 2-cocycle $b(g_1, g_2) \in Z^2(\mathfrak{g}, \mathbf{K})$ (not necessarily a coboundary). The classification of quasi-Frobenius subalgebras in $sl(n)$ can be found in [14]. In section 5 we shall show that extended and peripheric extended twists correspond to a class of Frobenius algebras.

The deformations of quantized algebras include the deformations of their Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$. The deformation properties both of $\mathfrak{g}$ and of $\mathfrak{g}^*$ must be taken into consideration. When a Lie algebra $\mathfrak{g}_1^*(\mu_1^*)$ (with composition μ_1^*) is deformed to first order:

$$(\mu_1^*)_t = \mu_1^* + t\mu_2^*$$

its deforming function μ_2^* is also a Lie product and the deformed property becomes reciprocal: μ_1^* can be considered as a first-order deforming function for algebra $\mathfrak{g}_2^*(\mu_2^*)$. Let $\mathfrak{g}(\mu)$ be a Lie algebra that form Lie bialgebras both with $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$. This means that we have a one-dimensional family $\{(\mathfrak{g}, (\mathfrak{g}_1^*)_t)\}$ of Lie bialgebras and correspondingly a one-dimensional family of quantum deformations $\{\mathcal{A}_t(\mathfrak{g}, (\mathfrak{g}_1^*)_t)\}$ [1]. This situation provides the possibility to construct in the set of Hopf algebras a smooth curve connecting quantizations of the type $\mathcal{A}(\mathfrak{g}, \mathfrak{g}_1^*)$ with those of $\mathcal{A}(\mathfrak{g}, \mathfrak{g}_2^*)$. Such smooth transitions can involve contractions provided $\mu_2^* \in B^2(\mathfrak{g}_1^*, \mathfrak{g}_1^*)$. This happens in the case of JT, ET and some other twists (see [15] and references therein).

3. Extended twists and their limits

In the construction of extended Jordanian twists suggested in [11] the carrier algebras of type $\mathbf{L}$ play a crucial role. These are solvable subalgebras with at least four generators. To study the limiting properties of the ETs let us write down this carrier algebra $\mathbf{L}$ in the general form

$$\begin{aligned} &[H, E] = \delta E \qquad [H, A] = \alpha A \qquad [H, B] = \beta B \\ &[A, B] = \gamma E \qquad [E, A] = [E, B] = 0 \end{aligned} \tag{3.1}$$

$$\alpha + \beta = \delta. \tag{3.2}$$

This parametrization does not describe the full orbit of $\mathbf{L}$, but presents the essential part of it with the preserved general structure of Lie compositions.

In this algebra one can successively perform two non-trivial twists. The first one corresponds to the carrier subalgebra $B(2)$ with generators H and E. It is called the Jordanian twist and has the twisting element [9]

$$\Phi_j = e^{H\otimes\sigma} \tag{3.3}$$

where

$$\sigma = \frac{1}{\delta}\ln(1 + \gamma E). \tag{3.4}$$

This twisting element is a solution of the factorized twist equations (see equations (2.5) and (2.6)). It transforms the Hopf algebra $U(\mathbf{L})$ into $U_j(\mathbf{L})$:

$$\begin{aligned} \Delta_j(H) &= H \otimes e^{-\delta\sigma} + 1 \otimes H \\ \Delta_j(A) &= A \otimes e^{\alpha\sigma} + 1 \otimes A \\ \Delta_j(B) &= B \otimes e^{\beta\sigma} + 1 \otimes B \\ \Delta_j(E) &= E \otimes e^{\delta\sigma} + 1 \otimes E. \end{aligned} \tag{3.5}$$

The Jordanian twist (3.3) can be extended [11] by the factors

$$\Phi_E = e^{A\otimes Be^{-\beta\sigma}} \tag{3.6}$$

or

$$\Phi_{E'} = e^{-B\otimes Ae^{-\alpha\sigma}}. \tag{3.7}$$

The element Φ_E is itself a solution of the general twist equation (2.3) for the algebra $U_j(\mathbf{L})$. After being twisted by Φ_E the algebra $U_j(\mathbf{L})$ transforms into $U_E(\mathbf{L})$ defined by

$$\begin{aligned}
\Delta_E(H) &= H \otimes e^{-\delta\sigma} + 1 \otimes H - \delta A \otimes Be^{-(\beta+\delta)\sigma} \\
\Delta_E(A) &= A \otimes e^{-\beta\sigma} + 1 \otimes A \\
\Delta_E(B) &= B \otimes e^{\beta\sigma} + e^{\delta\sigma} \otimes B \\
\Delta_E(E) &= E \otimes e^{\delta\sigma} + 1 \otimes E.
\end{aligned} \tag{3.8}$$

The compositions (3.1) and (3.8) with the condition (3.2) define the three-dimensional set $\mathcal{H}$ of Hopf algebras. All the *internal* points of this set correspond to the twisted algebras of the same general structure and the same properties. To obtain relations (3.8) we can also start with $U(\mathbf{L})$ and apply to it the extended twist $\mathcal{F}_E = \Phi_E \, \Phi_j$ (the composition of Φ_E and Φ_j). Note also that for non-zero values of parameters twists Φ_E and $\Phi_{E'}$ being applied to algebra $U_j(\mathbf{L})$ give the equivalent sets of Hopf algebras $U_E(\mathbf{L}) \approx U_{E'}(\mathbf{L})$. The corresponding equivalence map is generated by the substitution $(A, B, \alpha, \beta) \rightleftharpoons (B, -A, \beta, \alpha)$.

The situation changes when we consider the boundaries of the set $\mathcal{H}$. As we shall see the peripheric Hopf algebras (when they exist) are not only inequivalent to the initial one, but in some cases correspond to a new kind of extended twist with specific properties.

In the following five cases the results are trivial.

1. $\gamma \to 0$. The Jordanian twist is trivialized. The extensions become insignificant. They correspond to twisting by primitive elements of an abelian algebra. The carrier subalgebra is here two-dimensional Abelian and co-Abelian.

2. $\delta \to 0$; $\alpha = -\beta \neq 0$. In this case the divergences are inevitable in $\Delta_E(A)$ and in $\Delta_E(B)$. No limiting Hopf algebras in this boundary subset.

3. $\delta \to 0$ and $\alpha \to 0$, $\gamma \neq 0$. In such case β also goes to zero. The behaviour of these parameters can be coordinated so that the limiting Hopf algebra exists (in spite of the divergences of the Jordanian twisting element Φ_j. In this limit the carrier algebra $\mathbf{L}^{(3)} \equiv \lim_{\delta,\alpha\to 0} \mathbf{L}$ is the central extension of Heisenberg algebra formed by A, B and E. Put $\alpha = a\delta$, $\beta = b\delta$ (with $a + b = 1$) and let $\sigma_0 \equiv \ln(1 + \gamma E)$. The coproducts of the Hopf algebra $U_q(\mathbf{L}^{(3)})$ are defined by the relations

$$\begin{aligned}
\Delta_q(H) &= H \otimes e^{-\sigma_0} + 1 \otimes H \\
\Delta_q(A) &= A \otimes e^{a\sigma_0} + 1 \otimes A \\
\Delta_q(B) &= B \otimes e^{b\sigma_0} + 1 \otimes B \\
\Delta_q(E) &= E \otimes e^{\sigma_0} + 1 \otimes E.
\end{aligned} \tag{3.9}$$

Only the last three relations are essential, corresponding to some special case of Heisenberg algebra quantization. One can easily check that any group-like elements f_A, f_B, f'_A, f'_B and f_E depending on E can serve to construct the coalgebra

$$\begin{aligned}
\Delta_q(A) &= A \otimes f_A + f'_A \otimes A \\
\Delta_q(B) &= B \otimes f_B + f'_B \otimes B \\
\Delta_q(E) &= E \otimes f_E + 1 \otimes E
\end{aligned} \tag{3.10}$$

that will form a Hopf algebra with the Heisenberg Lie composition $[A, B] = \gamma E$ in two distinct cases:

$$f_A f_B = f_E \qquad \text{and} \qquad f'_A f'_B = 1 \tag{3.11}$$

or

$$f_A f_B = 1 \qquad \text{and} \qquad f'_A f'_B = f_E = 1 + \widetilde{\gamma} E. \tag{3.12}$$

Thus we have two classes of quantisations of Heisenberg algebra within the scope of the coalgebraic relations (3.10). The Hopf algebra $U_q(\mathbf{L}^{(3)})$ refers to the first (with $f'_A = f'_B = 1$ and $f_E = 1 + \gamma E$). In this case the extensions

$$\Phi_E = e^{A\otimes B f_B^{-1}} \tag{3.13}$$

and

$$\Phi_{E'} = e^{-B\otimes A f_A^{-1}} \tag{3.14}$$

exist and lead to the following quantizations of Heisenberg algebra:

$$\begin{aligned} &\Delta_{q,E}(A) = A \otimes f_B^{-1} + 1 \otimes A && \Delta_{q,E'}(A) = A \otimes f_A + f_E \otimes A \\ &\Delta_{q,E}(B) = B \otimes f_B + f_E \otimes B && \Delta_{q,E'}(B) = B \otimes f_A^{-1} + 1 \otimes B \\ &\Delta_{q,E}(E) = E \otimes f_E + 1 \otimes E && \Delta_{q,E}(E') = E \otimes f_E + 1 \otimes E. \end{aligned} \tag{3.15}$$

Note that $\Delta_q(H)$ containing only central elements is not touched by these extension twists (the same is seen above for $\Delta_q(E)$). Thus the only function of the twists that survive in this case is to bridge different classes of quantizations of Heisenberg algebras.

4. $\delta \to 0$ and $\beta \to 0$. This is identical to case 3.

5. $\delta \to 0$ and $\gamma \to 0$. In this limit the carrier algebra $\mathbf{L}^{(5)} \equiv \lim_{\delta,\gamma\to 0} \mathbf{L}$ is the central extension of the two-dimensional algebra $e(2)$ of plane motions. For the consistent behaviour of parameters the Jordanian twist survives in a form

$$\Phi_j^{(5)} = e^{H\otimes(\gamma/\delta)E}.$$

The corresponding deformation $U(\mathbf{L}^{(5)}) \xrightarrow{\Phi_j} U(\mathbf{L}_j^{(5)})$ amounts to a trivial quantization of $U(e(2))$ by a function of the central generator E. No additional transformations are produced by the extensions Φ_E or $\Phi_{E'}$.

Note that in cases 2, 3 and 4 the carrier algebra $\mathbf{L}$ loses the property of being Frobenius (see section 5 for more details).

There are two cases that provide non-trivial carrier algebras and twists:

(i) $\alpha \to 0$; $\beta = \delta$. Let us rewrite the corresponding carrier algebra relations:

$$\begin{aligned} &[H, E] = \delta E && [H, A] = 0 && [H, B] = \delta B \\ &[A, B] = \gamma E && [E, A] = [E, B] = 0. \end{aligned} \tag{3.16}$$

This is the limiting element of the sequence of algebras of the type (3.1), we shall denote it $\mathbf{L}^c$. It has rank 2 while all the other members of the sequence have rank 1. The twists survive in the limit with the twisting elements

$$\Phi_j = e^{H\otimes\sigma} \tag{3.17}$$

$$\Phi_P = e^{A\otimes B e^{-\beta\sigma}}. \tag{3.18}$$

The twisted algebra $U_j(\mathbf{L}^c)$ is the limit of the sequence of Hopf algebras defined by the coproducts (3.5):

$$\begin{aligned} \Delta_j(H) &= H \otimes e^{-\delta\sigma} + 1 \otimes H \\ \Delta_j(A) &= A \otimes 1 + 1 \otimes A \\ \Delta_j(B) &= B \otimes e^{\delta\sigma} + 1 \otimes B \\ \Delta_j(E) &= E \otimes e^{\delta\sigma} + 1 \otimes E. \end{aligned} \tag{3.19}$$

The second twisting element Φ_P does not depend on δ and leads to the algebra $U_P(\mathbf{L}^c)$ with the coproduct:

$$\begin{aligned}
\Delta_P(H) &= H \otimes e^{-\delta\sigma} + 1 \otimes H - \delta A \otimes Be^{-2\delta\sigma} \\
\Delta_P(A) &= A \otimes e^{-\delta\sigma} + 1 \otimes A \\
\Delta_P(B) &= B \otimes e^{\delta\sigma} + e^{\delta\sigma} \otimes B \\
\Delta_P(E) &= E \otimes e^{\delta\sigma} + 1 \otimes E.
\end{aligned} \tag{3.20}$$

The significant fact is that in $U_P(\mathbf{L}^c)$ the element $Be^{-\delta\sigma}$ is primitive. Together with the primitivity of A in $U_j(\mathbf{L}^c)$, this means that the twisting element Φ_P is now a solution of the factorized twist equations (2.5) and (2.6) contrary to the properties of the internal points of the set $\widehat{\mathcal{L}}$.

(ii) $\beta \to 0$; $\alpha = \delta$. Recall that in the general situation we have two possible extensions Φ_E and $\Phi_{E'}$ that give equivalent results. Here the picture is different. On the boundaries of $\widehat{\mathcal{L}}$ this degeneracy is removed and we are either to check both extensions for one type of limits or to study both limits for one of the extensions. This is the reason for considering this second limit separately.

The purely algebraic part $\mathbf{L}'^c$ looks like

$$\begin{aligned}
&[H, E] = \delta E \qquad [H, A] = \delta A \qquad [H, B] = 0 \\
&[A, B] = \gamma E \qquad [E, A] = [E, B] = 0
\end{aligned} \tag{3.21}$$

and its Jordanian twist $U_j(\mathbf{L}'^c)$

$$\begin{aligned}
\Delta_j(H) &= H \otimes e^{-\delta\sigma} + 1 \otimes H \\
\Delta_j(A) &= A \otimes e^{\delta\sigma} + 1 \otimes A \\
\Delta_j(B) &= B \otimes 1 + 1 \otimes B \\
\Delta_j(E) &= E \otimes e^{\delta\sigma} + 1 \otimes E
\end{aligned} \tag{3.22}$$

is still equivalent to the previous one, $U_j(\mathbf{L}^c)$ (see equations (3.19)). The extension of the JT has now a form that is essentially different to that of (3.18):

$$\Phi_{P'} = e^{A \otimes B}. \tag{3.23}$$

The final peripheric Hopf algebra $U_{P'}(\mathbf{L}'^c)$ is defined by the relations

$$\begin{aligned}
\Delta_{P'}(H) &= H \otimes e^{-\delta\sigma} + 1 \otimes H - \delta A \otimes Be^{-\delta\sigma} \\
\Delta_{P'}(A) &= A \otimes 1 + 1 \otimes A \\
\Delta_{P'}(B) &= B \otimes 1 + e^{\delta\sigma} \otimes B \\
\Delta_{P'}(E) &= E \otimes e^{\delta\sigma} + 1 \otimes E.
\end{aligned} \tag{3.24}$$

In this case the generator B is primitive in the intermediate algebra (3.22), while A becomes primitive after the extended twist. Thus it does not satisfy the ordinary factorized twist equations (2.5) and (2.6). Nevertheless, the relations valid for $\Phi_{P'}$

$$\begin{aligned}
(\Delta_{\mathcal{F}} \otimes \mathrm{id})\mathcal{F} &= \mathcal{F}_{13}\mathcal{F}_{23} \\
(\mathrm{id} \otimes \Delta)\mathcal{F} &= \mathcal{F}_{12}\mathcal{F}_{13}
\end{aligned} \tag{3.25}$$

describe the solution of the general twist equation (2.3) in our case because both tensor multipliers in $\Phi_{P'}$ depend each time on a single generator providing an additional commutativity for twisting elements in $H \otimes H \otimes H$-space. (Despite the visual similarity the equations (3.25) can not be referred to the inverse of the twisting element $\mathcal{F}$ due to the structure of the coproduct $\Delta_{\mathcal{F}}$.)

The universal $\mathcal{R}$-matrices have the form

$$\mathcal{R} = e^{Be^{-\delta\sigma}\otimes A} e^{\sigma\otimes H} e^{-H\otimes\sigma} e^{-A\otimes Be^{-\delta\sigma}} \tag{3.26}$$

in the first case, and

$$\mathcal{R} = e^{B\otimes A} e^{\sigma\otimes H} e^{-H\otimes\sigma} e^{-A\otimes B} \tag{3.27}$$

in the second. In both cases the deformation parameter can be introduced by the substitution $E \to \xi E$; $A \to \xi A$. This supplies the deformed algebra with the ordinary classical limit when $\xi \to 0$, and gives the possibility to write down the classical r-matrix. It has the same form in both cases:

$$r = A \wedge B + \frac{\gamma}{\delta} H \wedge E \tag{3.28}$$

(though defined for different carrier algebras (3.16) and (3.21)). Its form guarantees that in both cases the coboundary Lie bialgebras originating from it are self-dual.

Just as in the case of the extended Jordanian twist [11], one can append any number of similar extensions of type Φ_P (correspondingly $\Phi_{P'}$) to the initial Jordanian twist Φ_j for any number of pairs of equivalent eigenvectors (A_m, B_m) of the adjoint operator $\mathrm{ad}(H)$ and with the only non-zero commutators $[A_m, B_m] = \gamma E$.

4. Peripheric extended twists for simple Lie algebras—$sl(4)$ as an example

To demonstrate some other properties of the peripheric extended twists let us apply them to deform the universal envelopings of simple Lie algebras. The corresponding carrier subalgebras can be found in all the simple Lie algebras with rank no lower than 2. We shall work with the algebra $U(sl(4))$ in order to present a completely non-degenerate case. The canonical $gl(4)$ basis $\{E_{ij};\ i, j = 1, \ldots, 4\}$ will be used with commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}. \tag{4.1}$$

We shall study the PET with the carrier algebra $\mathbf{L}'^{c}$, which is of the second type (see equations (3.21)). Let us injected it into $sl(4)$ in the following way:

$$\begin{aligned} H &= E_{11} - E_{22} \equiv H_{12} \qquad & E &= E_{24} \\ A &= E_{23} & B &= E_{34}. \end{aligned} \tag{4.2}$$

This kind of injection corresponds to fixed values of the parameters

$$\alpha = \delta = -1 \qquad \gamma = 1 \qquad \beta = 0 \tag{4.3}$$

with

$$\sigma = -\ln(1 + E_{24}). \tag{4.4}$$

The universal enveloping algebra $U(sl(4))$ can be twisted by the PET

$$\mathcal{F}_{P'} = e^{E_{23}\otimes E_{34}} e^{H_{12}\otimes\sigma}. \tag{4.5}$$

The deformed algebra $U_{P'}(sl(4))$ thus obtained has comultiplications that are much less cumbersome compared with the result of an ordinary ET (see [11]):

$$\Delta_{P'}(H_{12}) = H_{12} \otimes e^{\sigma} + E_{23} \otimes E_{34} e^{\sigma} + 1 \otimes H_{12}$$

$$\Delta_{P'}(H_{13}) = H_{13} \otimes 1 + 1 \otimes H_{13}$$

$$
\begin{aligned}
\Delta_{P'}(H_{14}) &= H_{14} \otimes 1 + 1 \otimes H_{14} + H_{12} \otimes (1 - e^{\sigma}) - E_{23} \otimes E_{34}e^{\sigma} \\
\Delta_{P'}(E_{12}) &= E_{12} \otimes e^{2\sigma} - E_{13} \otimes E_{34}e^{2\sigma} + 1 \otimes E_{12} + H_{12} \otimes E_{14}e^{\sigma} + E_{23} \otimes E_{34}E_{14}e^{\sigma} \\
\Delta_{P'}(E_{13}) &= E_{13} \otimes e^{\sigma} + 1 \otimes E_{13} - E_{23} \otimes E_{14} \\
\Delta_{P'}(E_{14}) &= E_{14} \otimes e^{\sigma} + 1 \otimes E_{14} \\
\Delta_{P'}(E_{21}) &= E_{21} \otimes e^{-2\sigma} + 1 \otimes E_{21} \\
\Delta_{P'}(E_{23}) &= E_{23} \otimes 1 + 1 \otimes E_{23} \\
\Delta_{P'}(E_{24}) &= E_{24} \otimes e^{-\sigma} + 1 \otimes E_{24} \\
\Delta_{P'}(E_{31}) &= E_{31} \otimes e^{-\sigma} + 1 \otimes E_{31} + E_{21} \otimes E_{34}e^{-\sigma} \\
\Delta_{P'}(E_{32}) &= E_{32} \otimes e^{\sigma} + 1 \otimes E_{32} + H_{13} \otimes E_{34}e^{\sigma} \\
\Delta_{P'}(E_{34}) &= E_{34} \otimes 1 + 1 \otimes E_{34} + E_{24} \otimes E_{34} \\
\Delta_{P'}(E_{41}) &= E_{41} \otimes e^{-\sigma} + 1 \otimes E_{41} + E_{23} \otimes E_{31} - H_{12} \otimes E_{21}e^{\sigma} - E_{23} \otimes E_{34}E_{21}e^{\sigma} \\
\Delta_{P'}(E_{42}) &= E_{42} \otimes e^{\sigma} - E_{43} \otimes E_{34}e^{\sigma} + E_{23} \otimes E_{32} + 1 \otimes E_{42} \\
&\quad - H_{12} \otimes H_{24}e^{\sigma} + H_{12} \otimes (e^{2\sigma} - e^{\sigma}) - E_{23} \otimes H_{24}E_{34}e^{\sigma} \\
&\quad + E_{23} \otimes E_{34}(2e^{2\sigma} - e^{\sigma}) + H_{12}^2 \otimes (e^{2\sigma} - e^{\sigma}) \\
&\quad + 2H_{12}E_{23} \otimes E_{34}e^{2\sigma} + E_{23}^2 \otimes E_{34}^2 e^{2\sigma} - H_{12}E_{23} \otimes E_{34}e^{\sigma} \\
\Delta_{P'}(E_{43}) &= E_{43} \otimes 1 + 1 \otimes E_{43} + E_{23} \otimes H_{34} - H_{12} \otimes E_{23}e^{\sigma} \\
&\quad - E_{23} \otimes E_{34}E_{23}e^{\sigma} + H_{12}E_{23} \otimes E_{24}e^{\sigma} - E_{23}^2 \otimes E_{34}e^{\sigma}.
\end{aligned}
\tag{4.6}
$$

The following universal $\mathcal{R}$-matrix corresponds to this PET deformation:

$$\mathcal{R} = e^{\xi E_{34} \otimes E_{23}} e^{\sigma \otimes H_{12}} e^{-H_{12} \otimes \sigma} e^{-\xi E_{23} \otimes E_{34}}. \tag{4.7}$$

In this expression the deformation parameter has been introduced (see section 3), so here $\sigma = -\ln(1 + \xi E_{24})$. The corresponding classical r-matrix looks like

$$r = E_{34} \wedge E_{23} + H_{12} \wedge E_{24}. \tag{4.8}$$

5. Peripheric twists and Drinfeld–Jimbo quantizations

It has been known for a long time that some types of Jordanian quantizations can be treated as limiting structures for certain smooth sequences of standard deformations [9, 16, 17]. It was proved in [15] that this property is provided by the specific correlation between the Lie bialgebras of Drinfeld–Jimbo and ET quantizations.

Let $(\mathfrak{g}, \mathfrak{g}^*_{\mathrm{DJ}})$ and $(\mathfrak{g}, \mathfrak{g}^*_j)$ be the Lie bialgebras corresponding to Drinfeld–Jimbo and Jordanian quantizations of $\mathfrak{g}$, respectively. Let μ, μ^*_{DJ} and μ^*_j denote the corresponding Lie composition maps. It was demonstrated in [15] that if μ^*_j is a 2-coboundary for the Lie algebra $\mathfrak{g}^*_{\mathrm{DJ}}$, i.e.

$$\mu^*_j \in B^2(\mathfrak{g}^*_{\mathrm{DJ}}, \mathfrak{g}^*_{\mathrm{DJ}})$$

then in the set of deformation quantizations of $U(\mathfrak{g})$ there exists a smooth curve connecting $U_j(\mathfrak{g})$ (or in the analogous conditions $U_E(\mathfrak{g})$) with the standard deformation $U_{\mathrm{DJ}}(\mathfrak{g})$.

Smoothness is defined here in the topology very similar to the power series one (see [18, 19] for details).

It is important to know whether the algebras twisted by PETs can also be connected with DJ quantizations, thus describing the limiting cases with respect to the standard deformations. In the context of this problem we need the inverse of the previous statement. Let us formulate it as follows.

Lemma 1. *Let $U_A(\mathfrak{g})$ and $U_{A'}(\mathfrak{g})$ be two inequivalent quantum deformations of $U(\mathfrak{g})$ and $\mathcal{H}(p,q)$ be a smooth curve connecting them. If the curve has the properties:*
(i) $\mathcal{H}(p,q)_{p=0} = U_A(\mathfrak{g})$ and $\mathcal{H}(p,q)_{q=1} = U_{A'}(\mathfrak{g})$,
(ii) $\mathcal{H}(p,q)$ depends analytically on q,
then the Lie maps of algebras $\mathfrak{g}^*_A$ and $\mathfrak{g}^*_{A'}$ are the cocycles of each other:

$$\begin{aligned} &\mu^*_{A'} \in Z^2(\mathfrak{g}^*_A, \mathfrak{g}^*_A) \\ &\mu^*_A \in Z^2(\mathfrak{g}^*_{A'}, \mathfrak{g}^*_{A'}). \end{aligned} \tag{5.1}$$

Proof. The interior of the set of curves $\{\mathcal{H}(p,q),\ q \in [0, q_1]\}$ forms a neighbourhood $\mathcal{O}(\mathfrak{g})$ of $U(\mathfrak{g})$ (in the topology induced in the two-dimensional subset $\mathcal{H}(p,q)$). The parameters p and q are the natural coordinates in a map covering the neighbourhood $\mathcal{O}(\mathfrak{g})$. Thus, for a sufficiently small fixed $q_0 \in [0, q_1]$ and any small p the pair $\left(\mu\ , q_0\mu^*_A + p\mu^*_{A'} \equiv \mu^*_{q_0,p}\right)$ is a Lie bialgebra. This means that $\mu^*_{q_0,p}$ is the first-order deformation of $q_0\mu^*_A$. But $\mu^*_{A'}$ itself is a Lie algebra. So, $\mu^*_{q_0,p}$ is also the first-order deformation of $p\mu^*_{A'}$. □

The conditions imposed in lemma 1 are natural, they correspond to the supposition that there are no singularities in the neighbourhood of $U(\mathfrak{g})$ in the set of its deformation quantizations.

In the example we presented in section 4, the Lie map $\mu^*_{\mathrm{DJ}}(sl(4))$ of the algebra $(sl(4))^*_{\mathrm{DJ}}$ in the basis $\{X_{ik}\}$ canonically dual to $\{E_{ik}\}$ has the following non-zero commutators:

$$\begin{aligned} &[X_{ii}, X_{kl}]_{k\leqslant l} = \delta_{ik}X_{il} - \delta_{il}X_{ki} \\ &[X_{ii}, X_{kl}]_{k\geqslant l} = -\delta_{ik}X_{il} + \delta_{il}X_{ki} \\ &[X_{ij}, X_{kl}]_{i<j,\,k<l} = 2(\delta_{jk}X_{il} - \delta_{il}X_{kj}) \\ &[X_{ij}, X_{kl}]_{i>j,\,k>l} = -2(\delta_{jk}X_{il} - \delta_{il}X_{kj}). \end{aligned} \tag{5.2}$$

The Lie algebra $(sl(4))^*_{P'}$ corresponding to the PET performed by (4.5) can be extracted from the coproducts (4.6):

$$\begin{aligned} &[X_{11}, X_{14}] = X_{12} && [X_{11}, X_{21}] = -X_{41} \\ &[X_{11}, X_{22}] = -X_{42} && [X_{11}, X_{24}] = X_{22} - X_{44} \\ &[X_{11}, X_{23}] = -X_{43} && [X_{11}, X_{34}] = X_{32} \\ &[X_{11}, X_{44}] = X_{42} && [X_{22}, X_{21}] = X_{41} \\ &[X_{22}, X_{23}] = X_{43} && [X_{22}, X_{24}] = -X_{22} + X_{44} \\ &[X_{22}, X_{44}] = -X_{42} && [X_{33}, X_{23}] = -X_{43} \\ &[X_{33}, X_{34}] = -X_{32} && [X_{44}, X_{23}] = X_{43} \\ &[X_{12}, X_{24}] = -2X_{12} && [X_{13}, X_{24}] = -X_{13} \\ &[X_{13}, X_{34}] = -X_{12} && [X_{14}, X_{22}] = X_{12} \\ &[X_{14}, X_{23}] = X_{13} && [X_{14}, X_{24}] = -X_{14} \end{aligned}$$

$$\begin{array}{ll}
[X_{21}, X_{24}] = 2X_{21} & [X_{21}, X_{34}] = X_{31} \\
[X_{23}, X_{31}] = X_{41} & [X_{23}, X_{32}] = X_{42} \\
[X_{23}, X_{34}] = -X_{22} + X_{44} & [X_{24}, X_{31}] = -X_{31} \\
[X_{24}, X_{32}] = X_{32} & [X_{24}, X_{34}] = X_{34} \\
[X_{24}, X_{41}] = -X_{41} & [X_{24}, X_{42}] = X_{42} \\
[X_{34}, X_{43}] = X_{42}. &
\end{array} \tag{5.3}$$

We shall denote this set of compositions $\mu^*_{P'}(sl(4))$.

One can check by direct computations that the Lie multiplications $\mu^*_{\mathrm{DJ}}(sl(4))$ and $\mu^*_{P'}(sl(4))$ are not the first-order deformations of each other. This means (taking into account that they are themselves the Lie compositions) that they are not the 2-cocycles of each other. So the conditions (5.1) are not satisfied, and according to lemma 1 the Hopf algebras $U_{\mathrm{DJ}}(sl(4))$ and $U_{P'}(sl(4)))$ cannot be connected by a smooth curve. We have come to the conclusion that $U_{P'}(sl(4)))$ cannot be obtained from the Drinfeld–Jimbo deformation of $U(sl(4))$ by a contraction or by any other smooth limiting process. This feature clearly shows how different could be the results of quantum deformations by extended and by peripheric twists.

The facts discussed above are intimately connected with the problem of the equivalence of different CYBE solutions, and in this context with the properties of the corresponding quasi-Frobenius algebras. We have seen that all the algebras belonging to the set $\widetilde{\mathcal{L}} = \{\mathbf{L}(\alpha, \delta-\alpha, \gamma, \delta) | \gamma \neq 0, \delta \neq 0\}$ are at least quasi-Frobenius. This property can be summarized as follows.

Lemma 2. *All the elements of the set $\widetilde{\mathcal{L}}$ are Frobenius algebras.*

Proof. For all the algebras $\mathbf{L}$ of the set $\widetilde{\mathcal{L}}$ the form

$$b(g_1, g_2) = E^*([g_1, g_2]) \qquad g_1, g_2 \in \mathbf{L}$$

is non-degenerate. Here E^* is the functional canonically dual to the basic element $E \in \mathbf{L}$. □

Note that our results are in total agreement with the classification of quasi-Frobenius algebras of low dimension given by Stolin [14]. One can check that the set $\widetilde{\mathcal{L}}$ is equivalent to the class $\{P_{a_1,a_2,a_3} | a_1 \neq a_3\}$ (see [14, proposition 1.2.3]).

6. Conclusions

The peripheric twists described in this paper are not continuously connected with Drinfeld–Jimbo deformations despite the fact that the carrier subalgebras of the peripheric and ordinary extended twists belong to the same smooth family of Frobenius algebras. Taking into account that the $U_E(sl(n))$ algebra quantized by certain types of ET can be treated as the continuous limit of DJ deformations [15], we have at least the superposition of two smooth transitions that can connect DJ and PET deformations. In the case studied above the algebra $\mathbf{L}(\alpha, 0, \gamma, \delta) \subset sl(4)$ can be obtained from $\mathbf{L}(1, 1, 1, 2) \subset sl(3) \subset sl(4)$ by means of a 'rotation' in the space of the Cartan subalgebra of $sl(4)$. We want to stress that the 'rotation' connecting $\mathbf{L}(1, 1, 1, 2)$ with $\mathbf{L}(-1, 0, 1, -1)$ is not a similarity transformation for $\mathbf{L}$ and thus cannot be used to carry properties from the ET to the PET and vice versa. Nevertheless, it might also be possible to simulate analogous 'rotations' in the set of modified DJ deformations (using multiparametric quantizations or applying the continuous families of dual groups [20]). If both 'rotations' could be matched the possibility of a contraction-like smooth transition between modified DJ and PET deformations might exist.

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